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ON THE DECAY OF A NORMALLY DISTRIBUTED AND HOMOGENEOUS TURBULENT VELOCITY FIELD

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CONTENTS

This paper examines the dynamical behaviour of a field of homogeneous turbulence in which the joint-probability distribution of the fluctuating velocity components at three points is approximately normal. In principle, the analysis is formulated entirely in terms of the mean values

$$\overline{u_i u_j'} \quad \text{and} \quad \overline{u_i u_j' u_k''},$$

where the number of primes denotes the point at which the velocity components are taken. First, the kinematical properties of the three-point correlation are obtained by techniques similar to those used in the well-known theory of the two-point correlation. In the particular case of isotropic turbulence, the necessary extensions to the existing invariant theory lead to the result that the three-point correlation is completely defined by two scalar functions. Two independent dynamical relations between these correlations are then derived from the Navier-Stokes equation, and the remainder of the paper is based on this (determinate) system of equations. These remarks refer only to the principle of the calculations; in fact, most of the results are obtained in terms of the Fourier transforms of the correlations defined above.

The first set of deductions from the governing equations refer to the decay of isotropic turbulence at large Reynolds numbers. In particular, the exact solution of the inviscid equations for the vorticity is obtained, and it is shown to be consistent with the predictions of Kolmogoroff's theory of local similarity after a sufficiently long time of decay. The distribution of energy transfer between eddies of different sizes is also examined for a special form of the energy spectrum of turbulence, and the general features of this distribution appear to be satisfactory in the main energy-containing range of the spectrum.

The remaining results are concerned with energy transfer in the large eddies. It is shown, beyond all reasonable doubt, that the magnitude of this energy transfer is such that the large eddies are not permanent during decay. Immediate consequences of this result are that Loitsiansky's integral is not an invariant of the motion, and that the usual triple correlation function $k(r)$ is proportional to r^{-4} for large values of r. These conclusions are inconsistent with the theory initiated by Loitsiansky, and developed by Lin and Batchelor. The cause of this inconsistency is attributed to the dynamical unlikelihood of the basic assumptions made by these earlier authors.

General introduction

The hypothesis that the joint-probability distribution of fluctuating velocity components at two different points in homogeneous turbulence is approximately normal has played an important part in recent research in the subject. It has been used both in the form stated above and in an alternative form, the equivalence of which was pointed out by Batchelor (1951). In the alternative hypothesis, which was first made by Heisenberg (1948), it is supposed that the random Fourier coefficients of the velocity distribution are statistically independent. The second assumption is more special than the first, inasmuch as it implies† that the joint-probability distribution of velocity components taken simultaneously at any number of points in the turbulence is of the normal form. The analysis of the present paper involves the assumption that certain properties of the joint-probability distribution of the velocity components at *three* different points are of the normal form.

Since there is no *a priori* reason why a quantity that must satisfy the equations of continuity and motion should be distributed according to the normal law, it is necessary to appeal to experimental evidence for support for the hypothesis. Much of this has been collected together by Batchelor (1951). So far as they go, the experiments indicate that the Fourier coefficients of the velocity field at two different wave-numbers are approximately statistically independent, provided both these wave-numbers are small compared with a certain wave-number which itself appears to be sufficiently large to make the hypothesis reasonably accurate over the greater part of the energy spectrum of turbulence. Recent measurements by Uberoi (1953) are consistent with this conclusion. There is little reason for doubting Batchelor's comment on the necessary qualification to the hypothesis: it is that the motion of the smallest eddies is most strongly affected by the inertial interaction between different wave-numbers, and that the corresponding Fourier coefficients should therefore exhibit the greatest deviation from statistical independence.

In one or other of the forms indicated above, the hypothesis has been made the basis of a discussion of the statistical properties of the pressure field by Heisenberg (1948), by Obukhoff (1949) and by Batchelor (1951). On the other hand, the application of the hypothesis to the central problem of the decay of homogeneous turbulence has received comparatively little attention in the literature. Yet it was in this connexion that the approximation was first introduced into the subject by Millionshtchikov (1941 *a*, *b*). The details of Millionshtchikov's work, which was restricted to the small Reynolds number conditions pertaining to the final period of decay, need not concern us here, but the principle developed by him is relevant. It is that, although the mechanism of decay is intimately connected with the skewness of the joint-probability distribution of the velocity components at two points in the turbulence (a quantity which is zero for a normal distribution), the form of the equations of fluid motion is such that the mechanical development of this skewness may be calculated in detail if it is assumed that the relation between certain mean values involving

† Subject to certain mathematical conditions that are unlikely to cause trouble in the physical problem. See, for example, Batchelor (1953).

products of four and two velocity components is the one appropriate to a normal probability distribution. In this way, an approximate theory of decay may be developed, which takes into account the skewness whose physical significance is the all-important inertial exchange of energy between eddies of different sizes.

Thus, the favourable experimental evidence and the importance of the problem of decay together provide ample justification for examining the dynamical consequences of the hypothesis in question. Moreover, there is some interest to be attached to an examination of the problem of decay on the basis of a direct assumption about the fundamental probability distribution, as opposed to the more usual procedure of discussing the physical nature of the transfer of energy between eddies of different sizes. For the approach, being of a rather different type, may be expected to yield a different type of information and thus to further an understanding of the mechanics of homogeneous turbulence.

The purpose of the present contribution, therefore, is to derive the dynamical equations that follow from the hypothesis, and to present some properties of the solutions of these equations. Part I of the paper is concerned with the formal derivation of the dynamical equations and includes the necessary extensions to the existing kinematical analysis. In part II, some features of the solutions of these equations are discussed which correspond to the decay of isotropic turbulence at large Reynolds numbers. Among the results obtained in this part of the paper are some which indicate that the existing theory of the motion of the largest eddies is inadequate. Consequently, part III contains a re-examination of the general problem of energy transfer at small wave-numbers, mainly on the basis of the special type of probability distribution assumed throughout the earlier parts, but also, to some extent, for general fields of homogeneous turbulence.

PART I. GENERAL KINEMATICS AND DYNAMICS

1. Kinematical relations in homogeneous turbulence

Consider a homogeneous turbulent motion of an incompressible fluid, in which the mean velocity is zero. If $\mathbf{u}(\mathbf{x})$ denotes the velocity at the point $\mathbf{x}$, then the condition of statistical homogeneity requires that the velocity-product mean values

$$R_{ij}(\mathbf{r}) = \overline{u_i(\mathbf{x})\, u_j(\mathbf{x}+\mathbf{r})} \tag{1}$$

and

$$R_{ijk}(\mathbf{r}, \mathbf{r}') = \overline{u_i(\mathbf{x})\, u_j(\mathbf{x}+\mathbf{r})\, u_k(\mathbf{x}+\mathbf{r}')} \tag{2}$$

should be functions only of the variables indicated.† In a discussion of the dynamical properties of the tensors (1) and (2), it is necessary to consider mean values of certain products involving the pressure. Thus, if $p(\mathbf{x})$ denotes the deviation of the pressure at the point $\mathbf{x}$ from its uniform mean value, and ρ denotes the uniform density of the fluid, these new tensors are

$$P_i(\mathbf{r}) = \frac{1}{\rho}\overline{p(\mathbf{x})\, u_i(\mathbf{x}+\mathbf{r})} \tag{3}$$

and

$$P_{ij}(\mathbf{r}, \mathbf{r}') = \frac{1}{\rho}\overline{p(\mathbf{x})\, u_i(\mathbf{x}+\mathbf{r})\, u_j(\mathbf{x}+\mathbf{r}')}, \tag{4}$$

† And, of course, of the time. Throughout the paper, physical quantities under a mean-value sign are always taken at the same instant, and explicit reference to the time in this context is omitted.

which, by virtue of the homogeneity, are again functions only of the variables indicated. In the boundary-free problem of homogeneous turbulence, the pressure field is determined uniquely by the velocity field, but the relationship is cumbersome, and, for kinematical purposes, it is more convenient to introduce new symbols for the tensors (3) and (4) and to regard them as independent of tensors of the type (1) and (2). At a later stage, these tensors involving the pressure will be eliminated from the analysis.

All these tensors satisfy certain symmetry conditions, obtained by permuting the products under the mean-value sign. Thus, corresponding to the well-known condition

$$R_{ij}(\mathbf{r}) = R_{ji}(-\mathbf{r}), \tag{5}$$

the third-order tensor $R_{ijk}(\mathbf{r},\mathbf{r}')$ satisfies the cyclic condition

$$R_{ijk}(\mathbf{r},\mathbf{r}') = R_{jki}(\mathbf{r}'-\mathbf{r}, -\mathbf{r}), \tag{6}$$

and the acyclic condition
$$R_{ijk}(\mathbf{r},\mathbf{r}') = R_{ikj}(\mathbf{r}',\mathbf{r}). \tag{7}$$

Similarly, the tensor (4) satisfies the condition

$$P_{ij}(\mathbf{r},\mathbf{r}') = P_{ji}(\mathbf{r}',\mathbf{r}). \tag{8}$$

The incompressibility condition
$$\frac{\partial u_i}{\partial x_i} = 0 \tag{9}$$

yields the further result that each tensor is solenoidal in every index. Thus, for the velocity-product mean values,

$$\frac{\partial}{\partial r_i} R_{ij}(\mathbf{r}) = \frac{\partial}{\partial r_j} R_{ij}(\mathbf{r}) = 0 \tag{10}$$

and
$$\left(\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i'}\right) R_{ijk}(\mathbf{r},\mathbf{r}') = \frac{\partial}{\partial r_j} R_{ijk}(\mathbf{r},\mathbf{r}') = \frac{\partial}{\partial r_k'} R_{ijk}(\mathbf{r},\mathbf{r}') = 0, \tag{11}$$

and for mean values involving the pressure,

$$\frac{\partial}{\partial r_i} P_i(\mathbf{r}) = 0 \tag{12}$$

and
$$\frac{\partial}{\partial r_i} P_{ij}(\mathbf{r},\mathbf{r}') = \frac{\partial}{\partial r_j'} P_{ij}(\mathbf{r},\mathbf{r}') = 0. \tag{13}$$

By virtue of the symmetry conditions (5) to (8), only one of the solenoidal conditions in each line is independent; but when deriving the forms of these tensors appropriate to isotropic turbulence, it is usually more convenient to impose all the solenoidal conditions before considering the further consequences of symmetry.

Now, it has already been noted that the fundamental approximation to be made in the dynamical equations is one that is not equally good in all eddy sizes. More precisely, the approximation is only reasonable for those Fourier components of the velocity whose wave-number is small compared with some empirically determined value. There is therefore good reason for dealing with the Fourier transforms of the tensors (1) to (4), since by so doing, careful account may be kept of those regions of the spectrum in which the theory is likely to be applicable. Moreover, the transformation to Fourier space introduces greater mathematical tractability, inasmuch as spatial differential operators transform into algebraic operators. This last simplification is particularly important for tensor functions of more than one vector variable, such as are considered in this paper.

Accordingly, corresponding to the well-known spectral tensor

$$\Phi_{ij}(\mathbf{k}) = (2\pi)^{-3}\int R_{ij}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{r}, \tag{14}$$

the six-dimensional Fourier transform of $R_{ijk}(\mathbf{r},\mathbf{r}')$ may be defined by the equation

$$\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \mathrm{i}(2\pi)^{-6}\iint R_{ijk}(\mathbf{r},\mathbf{r}')\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}.\mathbf{r}+\mathbf{k}'.\mathbf{r}')}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'. \tag{15}$$

In these equations, $\mathrm{i} = \surd(-1)$, and the integrations are over all the spaces of the $\mathbf{r}$'s. Similarly, Fourier transforms of the tensors involving the pressure may be defined by the equations

$$\Pi_i(\mathbf{k}) = \mathrm{i}(2\pi)^{-3}\int P_i(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}.\mathbf{r}}\,\mathrm{d}\mathbf{r} \tag{16}$$

and

$$\Pi_{ij}(\mathbf{k},\mathbf{k}') = (2\pi)^{-6}\iint P_{ij}(\mathbf{r},\mathbf{r}')\,\mathrm{e}^{-\mathrm{i}(\mathbf{k}.\mathbf{r}+\mathbf{k}'.\mathbf{r}')}\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{r}'. \tag{17}$$

All these spectral tensors exist if the moduli of the respective integrands are integrable, i.e. if the tendency to statistical independence of conditions at large separations is sufficiently rapid. This will be assumed to be the case.

The tensors occurring in the integrands of equations (14) to (17) represent the mean values of products of physical quantities, and are consequently real. Hence the spectral tensors are complex quantities with simple properties of the type

$$\Phi^*_{ij}(\mathbf{k}) = \Phi_{ij}(-\mathbf{k}) \tag{18}$$

and

$$\Phi^*_{ijk}(\mathbf{k},\mathbf{k}') = \Phi_{ijk}(-\mathbf{k},-\mathbf{k}'), \tag{19}$$

where the asterisk denotes a complex conjugate.

The transformed versions of the symmetry conditions (5) to (7) follow immediately from the definitions of the spectral tensors. Thus, corresponding to the well-known relation

$$\Phi_{ij}(\mathbf{k}) = \Phi_{ji}(-\mathbf{k}), \tag{20}$$

which, in conjunction with (18), shows that $\Phi_{ij}(\mathbf{k})$ possesses Hermitian symmetry, the third-order tensor $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ must satisfy the conditions

$$\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \Phi_{jki}(\mathbf{k}',-\mathbf{k}-\mathbf{k}') \tag{21}$$

and

$$\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \Phi_{ikj}(\mathbf{k}',\mathbf{k}). \tag{22}$$

It may be noted that the rearrangement of wave-numbers in (21) and (22) is quite different from that in (19), so that the third-order spectral tensor does not possess any specially simple form by virtue of these conditions.

The solenoidal conditions (10) to (13), resulting from the equation of continuity, transform into conditions of orthogonality in Fourier space. Thus,

$$k_i\,\Phi_{ij}(\mathbf{k}) = k_j\,\Phi_{ij}(\mathbf{k}) = 0, \tag{23}$$

$$(k_i+k'_i)\,\Phi_{ijk}(\mathbf{k},\mathbf{k}') = k_j\,\Phi_{ijk}(\mathbf{k},\mathbf{k}') = k'_k\,\Phi_{ijk}(\mathbf{k},\mathbf{k}') = 0, \tag{24}$$

$$k_i\,\Pi_i(\mathbf{k}) = 0, \tag{25}$$

and

$$k_i\,\Pi_{ij}(\mathbf{k},\mathbf{k}') = k'_j\,\Pi_{ij}(\mathbf{k},\mathbf{k}') = 0. \tag{26}$$

In a later section, the isotropic forms are derived of tensors satisfying orthogonality conditions of this type.

The task is now to construct the dynamical equations for $\Phi_{ij}(\mathbf{k})$ and $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$, making use of the above geometrical simplifications resulting from the conditions of homogeneity and continuity.

2. Dynamical equations in homogeneous turbulence

The equations for the rates of change with time of the velocity-product mean values $R_{ij}(\mathbf{r})$ and $R_{ijk}(\mathbf{r},\mathbf{r}')$ may be obtained by the methods first developed by von Kármán & Howarth (1938) for isotropic turbulence. The dynamical equations for the spectral tensors $\Phi_{ij}(\mathbf{k})$ and $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ then follow by Fourier transformation.†

Thus, in the case of $R_{ij}(\mathbf{r})$,

$$\frac{\partial}{\partial t}R_{ij}(\mathbf{r}) = \frac{\partial}{\partial t}\overline{u_i(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})} = \mathrm{O}_2\left\{\overline{\frac{\partial u_i(\mathbf{x})}{\partial t}\,u_j(\mathbf{x}+\mathbf{r})}\right\}, \tag{27}$$

where the operator O_2 acting on a second-order tensor $C_{ij}(\mathbf{r})$ is such that

$$\mathrm{O}_2\{C_{ij}(\mathbf{r})\} = C_{ij}(\mathbf{r}) + C_{ji}(-\mathbf{r}). \tag{28}$$

Substitution for $\partial u_i(\mathbf{x})/\partial t$ from the Navier-Stokes equation

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_l}(u_i u_l) - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i \tag{29}$$

then gives

$$\frac{\partial}{\partial t}R_{ij}(\mathbf{r}) = \mathrm{O}_2\left\{\frac{\partial}{\partial r_l}R_{ijl}(\mathbf{r},0) + \frac{\partial}{\partial r_i}P_j(\mathbf{r}) + \nu\frac{\partial^2}{\partial r_l^2}R_{ij}(\mathbf{r})\right\}. \tag{30}$$

Since the operator O_2 is invariant with respect to a Fourier transformation, the transformed version of (30) may be written

$$\frac{\partial}{\partial t}\Phi_{ij}(\mathbf{k}) = \mathrm{O}_2\left\{k_l\int\Phi_{ijl}(\mathbf{k},\mathbf{k}')\,\mathrm{d}\mathbf{k}' - k_i\,\Pi_j(\mathbf{k}) - \nu k^2\Phi_{ij}(\mathbf{k})\right\}, \tag{31}$$

where $k = |\,\mathbf{k}\,|$. In the present notation, this is the equation first given by Batchelor (1949). The tensor $\Pi_j(\mathbf{k})$, which is derived from product mean values involving the pressure, may be eliminated from (31) by making use of the incompressibility condition. Thus, multiplying (31) by k_i, and using the orthogonality relations (23) and (25), one obtains

$$k^2\Pi_j(\mathbf{k}) = k_i k_l\int\Phi_{ijl}(\mathbf{k},\mathbf{k}')\,\mathrm{d}\mathbf{k}', \tag{32}$$

so that (31) becomes

$$\frac{\partial}{\partial t}\Phi_{ij}(\mathbf{k}) = \mathrm{O}_2\left\{k_l\left(\delta_{i\alpha} - \frac{k_i k_\alpha}{k^2}\right)\int\Phi_{\alpha jl}(\mathbf{k},\mathbf{k}')\,\mathrm{d}\mathbf{k}' - \nu k^2\Phi_{ij}(\mathbf{k})\right\}. \tag{33}$$

Equation (33) is the first of two dynamical relations between the second- and third-order spectral tensors.

† These equations can, of course, be obtained directly from the dynamical equation for a single Fourier coefficient. The present technique of transforming the equation in physical space is used in order to provide a record of the equations in either mode of representation.

A similar procedure may be employed for $R_{ijk}(\mathbf{r},\mathbf{r}')$ and $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$. From the definition (2) of $R_{ijk}(\mathbf{r},\mathbf{r}')$, we have

$$\frac{\partial}{\partial t}R_{ijk}(\mathbf{r},\mathbf{r}') = \mathrm{O}_3\left\{\overline{\frac{\partial u_i(\mathbf{x})}{\partial t}u_j(\mathbf{x}+\mathbf{r})\,u_k(\mathbf{x}+\mathbf{r}')}\right\}, \tag{34}$$

where the operator O_3 acting on a third-order tensor $C_{ijk}(\mathbf{r},\mathbf{r}')$ is such that

$$\mathrm{O}_3\{C_{ijk}(\mathbf{r},\mathbf{r}')\} = C_{ijk}(\mathbf{r},\mathbf{r}')+C_{jki}(\mathbf{r}'-\mathbf{r},\,-\mathbf{r})+C_{kij}(-\mathbf{r}',\mathbf{r}-\mathbf{r}'). \tag{35}$$

With the Navier-Stokes equation, (34) then becomes

$$\frac{\partial}{\partial t}R_{ijk}(\mathbf{r},\mathbf{r}') = \mathrm{O}_3\left\{\left(\frac{\partial}{\partial r_l}+\frac{\partial}{\partial r_l'}\right)\overline{u_i(\mathbf{x})\,u_l(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})\,u_k(\mathbf{x}+\mathbf{r}')}\right.$$
$$\left.+\left(\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_i'}\right)P_{jk}(\mathbf{r},\mathbf{r}')+\nu\left(\frac{\partial}{\partial r_l}+\frac{\partial}{\partial r_l'}\right)^2R_{ijk}(\mathbf{r},\mathbf{r}')\right\}. \tag{36}$$

The appearance of fourth-order velocity-product mean values on the right of equation (36) is a reflexion, in the present analysis, of the central difficulty of the subject. It is this difficulty, namely, that the form of the Navier-Stokes equation is such that any finite system of equations for mean values of fluctuating quantities necessarily contains more unknowns than there are equations, which has been the subject of so much discussion since the early days of the mixing-length theories. All† contributions to this aspect of the problem have related, explicitly or otherwise, product mean values of different order for the purpose of making the dynamical problem determinate. As has already been indicated, the present purpose is to examine the consequences of the hypothesis that fourth- and second-order velocity-product mean values are related in the manner appropriate to a normal probability distribution. Equation (36) shows that the specific probability distribution concerning which this assumption must be made is the joint-probability distribution of the velocity vector at three different points in the turbulence. Thus, writing

$$\overline{u_i(\mathbf{x})\,u_l(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})\,u_k(\mathbf{x}+\mathbf{r}')} = \overline{u_i(\mathbf{x})\,u_l(\mathbf{x})}\,.\,\overline{u_j(\mathbf{x}+\mathbf{r})\,u_k(\mathbf{x}+\mathbf{r}')}$$
$$+\overline{u_i(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})}\,.\,\overline{u_l(\mathbf{x})\,u_k(\mathbf{x}+\mathbf{r}')}$$
$$+\overline{u_i(\mathbf{x})\,u_k(\mathbf{x}+\mathbf{r}')}\,.\,\overline{u_l(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})}, \tag{37}$$

equation (36) becomes

$$\frac{\partial}{\partial t}R_{ijk}(\mathbf{r},\mathbf{r}') = \mathrm{O}_3\left\{\left(\frac{\partial}{\partial r_l}+\frac{\partial}{\partial r_l'}\right)[R_{ij}(\mathbf{r})\,R_{lk}(\mathbf{r}')+R_{ik}(\mathbf{r}')\,R_{lj}(\mathbf{r})]\right.$$
$$\left.+\left(\frac{\partial}{\partial r_i}+\frac{\partial}{\partial r_i'}\right)P_{jk}(\mathbf{r},\mathbf{r}')+\nu\left(\frac{\partial}{\partial r_l}+\frac{\partial}{\partial r_l'}\right)^2R_{ijk}(\mathbf{r},\mathbf{r}')\right\}. \tag{38}$$

With regard to the Fourier transformation of (38), it may be noted that if $\Gamma_{ijk}(\mathbf{k},\mathbf{k}')$ is the Fourier transform of $C_{ijk}(\mathbf{r},\mathbf{r}')$, in the sense of equation (15), then the Fourier transform of $\mathrm{O}_3\{C_{ijk}(\mathbf{r},\mathbf{r}')\}$ is

$$\Omega_3\{\Gamma_{ijk}(\mathbf{k},\mathbf{k}')\} = \Gamma_{ijk}(\mathbf{k},\mathbf{k}')+\Gamma_{jki}(\mathbf{k}',\,-\mathbf{k}-\mathbf{k}')+\Gamma_{kij}(-\mathbf{k}-\mathbf{k}',\mathbf{k}). \tag{39}$$

† Except for the treatment by Hopf (1952), in which the dynamical problem of turbulence is considered in its entirety.

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Hence the transformed version of (38) may be written

$$\frac{\partial}{\partial t}\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \Omega_3\{-(k_l+k_l')\,[\Phi_{ij}(\mathbf{k})\,\Phi_{lk}(\mathbf{k}')+\Phi_{ik}(\mathbf{k}')\,\Phi_{lj}(\mathbf{k})] \\ -(k_i+k_i')\,\Pi_{jk}(\mathbf{k},\mathbf{k}')-\nu(k_l+k_l')^2\,\Phi_{ijk}(\mathbf{k},\mathbf{k}')\}. \qquad (40)$$

From this equation, the pressure, in the form of the tensor $\Pi_{jk}(\mathbf{k},\mathbf{k}')$, must again be eliminated. Multiplying the equation by k_i+k_i', and using the orthogonality conditions (24) and (26), one finds†

$$(k_i+k_i')^2\,\Pi_{jk}(\mathbf{k},\mathbf{k}') = -(k_i+k_i')\,(k_l+k_l')\,[\Phi_{ij}(\mathbf{k})\,\Phi_{lk}(\mathbf{k}')+\Phi_{ik}(\mathbf{k}')\,\Phi_{lj}(\mathbf{k})], \qquad (41)$$

so that equation (40) becomes

$$\frac{\partial}{\partial t}\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \Omega_3\left\{-(k_l+k_l')\left[\delta_{i\alpha}-\frac{(k_i+k_i')\,(k_\alpha+k_\alpha')}{(k_m+k_m')^2}\right][\Phi_{\alpha j}(\mathbf{k})\,\Phi_{lk}(\mathbf{k}')+\Phi_{\alpha k}(\mathbf{k}')\,\Phi_{lj}(\mathbf{k})] \\ -\nu(k_l+k_l')^2\,\Phi_{ijk}(\mathbf{k},\mathbf{k}')\right\}. \qquad (42)$$

For given initial conditions, the equations (33) and (42) now serve to determine the unknown tensors $\Phi_{ij}(\mathbf{k})$ and $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ at subsequent times, over the range of wave numbers in which the experiments show the underlying approximation to be reasonably accurate. However, there remains the problem of how to obtain detailed results from these equations, bearing in mind the complexity of the tensor algebra. In this connexion, and also for its practical value, it is clearly important that the consequences of statistical isotropy should be examined,‡ for the tensor equations then reduce to a system of relatively few scalar equations. Accordingly, the isotropic forms of the various tensors and equations are considered in the following sections.

3. Isotropic spectral tensors

The isotropic form of tensor functions of vector arguments may be found by the method, based on a use of invariant theory, introduced by Robertson (1940). Moreover, since isotropy in physical space implies isotropy in Fourier space, the method may be applied directly to the Fourier transforms of isotropic tensors. Thus, the well-known result for a second-order tensor function of a single wave-number vector is

$$\phi_{\alpha\beta}(\mathbf{k}) = \phi_1 k_\alpha k_\beta + \phi_2 \delta_{\alpha\beta}, \qquad (43)$$

where the scalar coefficients ϕ_1 and ϕ_2 of the fundamental isotropic tensors are arbitrary functions of k^2. Similarly, the result for a third-order tensor function of two wave-number vectors is easily found to be

$$\phi_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}') = \phi_1 k_\alpha k_\beta k_\gamma + \phi_2 k_\alpha k_\beta k_\gamma' + \phi_3 k_\alpha k_\beta' k_\gamma + \phi_4 k_\alpha' k_\beta k_\gamma \\ +\phi_5 k_\alpha' k_\beta' k_\gamma' + \phi_6 k_\alpha' k_\beta' k_\gamma + \phi_7 k_\alpha' k_\beta k_\gamma' + \phi_8 k_\alpha k_\beta' k_\gamma' \\ +\phi_9 k_\alpha \delta_{\beta\gamma} + \phi_{10} k_\beta \delta_{\gamma\alpha} + \phi_{11} k_\gamma \delta_{\alpha\beta} \\ +\phi_{12} k_\alpha' \delta_{\beta\gamma} + \phi_{13} k_\beta' \delta_{\gamma\alpha} + \phi_{14} k_\gamma' \delta_{\alpha\beta}, \qquad (44)$$

† The resulting equation is the irrotational part of the dynamical equation, a special case of which yields the pressure correlations already fully discussed by the authors referred to in the General Introduction.

‡ This is not to say that the approximation of a normal probability distribution is any more valid in isotropic turbulence than in general homogeneous turbulence. Indeed, it seems difficult to find an argument against the existence of normal probability distributions in the more general case which does not apply equally well to isotropic turbulence. The point is more to examine the theory first in the mathematically simplest form.

where the defining scalars $\phi_1, \phi_2, \ldots$ are now arbitrary functions of the three variables $k^2, k'^2 = \mathbf{k}'.\mathbf{k}'$, and $\mathbf{k}.\mathbf{k}'$. Of course, any three combinations of these latter quantities may be taken as independent variables. From the physical standpoint, the set k^2, k'^2 and $k''^2 = \mathbf{k}''.\mathbf{k}''$, where

$$\mathbf{k}+\mathbf{k}'+\mathbf{k}'' = 0, \tag{45}$$

is probably the most significant, in view of the way in which tensors of this kind appear in the dynamical equation (42).

As is well known, the principal difficulty in the kinematical analysis of isotropic tensors arises not in finding the general forms (43) and (44), but in finding the further simplifications that follow from the equation of continuity. In physical space this requires a discussion of tensors that are solenoidal, and in Fourier space the task is to find tensors satisfying the orthogonality conditions (23) and (24). In either case, the number of independent scalar functions required for the definition of such tensors is considerably reduced. Thus, direct substitution of the orthogonality conditions (23) and (24) into the general isotropic forms (43) and (44) leads to a set of algebraic relations between the general defining scalars from which the required number of independent scalars may be derived. Excepting the specially simple case of $\Phi_{ij}(\mathbf{k})$, however, the derivation from these relations of the precise way in which the independent scalars appear as coefficients of the fundamental isotropic tensors is somewhat troublesome, and it is therefore considerably easier to use the alternative approach described below, in which tensors are produced directly whose defining scalars are in a convenient form and such that the relevant orthogonality conditions are satisfied identically.†

Thus, consider the tensor

$$\Delta_i^\alpha(\mathbf{k}) = \delta_{i\alpha} - k_i k_\alpha / k^2. \tag{46}$$

This is clearly orthogonal with respect to $\mathbf{k}$ in the index i (i.e. $k_i \Delta_i^\alpha(\mathbf{k}) = 0$), and also isotropic, since it is of the general form (43). Hence, if $\phi_{\alpha jk\ldots}(\mathbf{k}, \mathbf{k}', \ldots)$ is a general isotropic tensor of any order, and a function of any number of wave-number vectors, then the tensor

$$\Phi_{ijk\ldots}(\mathbf{k}, \mathbf{k}', \ldots) = \Delta_i^\alpha(\mathbf{k})\, \phi_{\alpha jk\ldots}(\mathbf{k}, \mathbf{k}', \ldots) \tag{47}$$

is a similar tensor, but with the further property that it is orthogonal with respect to $\mathbf{k}$ in the index i. Moreover, the general tensor with this orthogonality property may be written in the form (47), since it is always possible to choose the primary tensor $\phi_{\alpha jk\ldots}(\mathbf{k}, \mathbf{k}', \ldots)$ as the most general tensor which is orthogonal with respect to $\mathbf{k}$ in the index α. The second term in the definition of $\Delta_i^\alpha(\mathbf{k})$ then makes no contribution and (47) reduces to an identity. In other words, $\Delta_i^\alpha(\mathbf{k})$ is an operator which selects from a general isotropic tensor that part of it which satisfies the orthogonality condition in question. Repeated use of this operator therefore constitutes a rapid and direct method of constructing isotropic tensors with several orthogonality properties such as are encountered in turbulence theory.‡

† The general idea of constructing tensors which satisfy the continuity condition identically is largely due to Chandrasekhar (1950). The technique to be described may therefore be regarded as an extension of Chandrasekhar's results.

‡ It is worth noting that the technique represented by (47) is also applicable to the more general case of homogeneous turbulence. Moreover, the corresponding analysis in physical space, in terms of the operator $D_i^\alpha(\mathbf{r}) = \delta_{i\alpha}\nabla^2 - \partial^2/\partial r_i\, \partial r_\alpha$, appears to offer no intrinsic difficulty, especially if the operator $\partial/\partial r_i$ is adopted as the fundamental vector instead of the vector r_i used by Robertson (1940).

As an illustration, the second-order tensor $\Phi_{ij}(\mathbf{k})$, satisfying the conditions (23), may be derived from a primary tensor in the manner

$$\Phi_{ij}(\mathbf{k}) = \Delta_i^\alpha(\mathbf{k})\,\Delta_j^\beta(\mathbf{k})\,\phi_{\alpha\beta}(\mathbf{k}).$$

In this equation, $\phi_{\alpha\beta}(\mathbf{k})$ is a general isotropic tensor of the form (43). However, the term in $k_\alpha k_\beta$ in (43) vanishes under the operator $\Delta_i^\alpha(\mathbf{k})\,\Delta_j^\beta(\mathbf{k})$, so that the equation reduces to the well-known form

$$\Phi_{ij}(\mathbf{k}) = \Delta_i^\alpha(\mathbf{k})\,\Delta_j^\beta(\mathbf{k})\,\phi_1\,\delta_{\alpha\beta} = \frac{E(k)}{4\pi k^2}\,(\delta_{ij}-k_ik_j/k^2), \tag{48}$$

where the defining scalar is written in terms of the usual energy-spectrum function $E(k)$.

Consider now the tensor $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$, which is required to satisfy the three orthogonality conditions (24). This may be written in the manner

$$\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \Delta_i^\alpha(\mathbf{k}'')\,\Delta_j^\beta(\mathbf{k})\,\Delta_k^\gamma(\mathbf{k}')\,\phi_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}'), \tag{49}$$

where the primary tensor must be of the form (44). However, eight of the fourteen fundamental isotropic tensors in equation (44) vanish identically under one or other of the operators $\Delta_j^\beta(\mathbf{k})$ and $\Delta_k^\gamma(\mathbf{k}')$, leaving only

$$k_\alpha\delta_{\beta\gamma},\quad k'_\alpha\delta_{\beta\gamma},\quad k'_\beta\delta_{\gamma\alpha},\quad k_\gamma\delta_{\alpha\beta},\quad k_\alpha k'_\beta k_\gamma\quad\text{and}\quad k'_\alpha k'_\beta k_\gamma.$$

Further,

$$\Delta_i^\alpha(\mathbf{k}'')\,(k_\alpha\delta_{\beta\gamma}+k'_\alpha\delta_{\beta\gamma}) = 0$$

and

$$\Delta_i^\alpha(\mathbf{k}'')\,(k_\alpha k'_\beta k_\gamma+k'_\alpha k'_\beta k_\gamma) = 0,$$

so that two more tensors may be rejected,† which we choose as

$$k'_\alpha\delta_{\beta\gamma}\quad\text{and}\quad k'_\alpha k'_\beta k_\gamma.$$

Thus only four of the defining scalars of $\phi_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}')$ make a contribution to the orthogonal tensor $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$, and the remaining scalars may be taken as zero. Hence (49) becomes

$$\Phi_{ijk}(\mathbf{k},\mathbf{k}') = \Delta_i^\alpha(\mathbf{k}'')\,\Delta_j^\beta(\mathbf{k})\,\Delta_k^\gamma(\mathbf{k}')\,[\Phi k_\alpha\delta_{\beta\gamma}+\Phi_1 k'_\beta\delta_{\gamma\alpha}+\Phi_2 k_\gamma\delta_{\alpha\beta}+\Psi k_\alpha k'_\beta k_\gamma], \tag{50}$$

where the notation for the four scalars has been changed from that in (44).

So far, no account has been taken of the symmetry conditions (20), (21) and (22). In the case of $\Phi_{ij}(\mathbf{k})$, (20) yields no further information, since an isotropic second-order tensor is *ipso facto* symmetric. But further relations between the defining scalars of $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ follow from these conditions. By virtue of the symmetry of the product

$$\Delta_i^\alpha(\mathbf{k}'')\,\Delta_j^\beta(\mathbf{k})\,\Delta_k^\gamma(\mathbf{k}'),$$

it is clear that the conditions (21) and (22) on $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ impose identical conditions on $\phi_{\alpha\beta\gamma}(\mathbf{k},\mathbf{k}')$. Hence we must have

$$\left.\begin{aligned}\Phi(k,k',k'') &= -\Phi(k',k,k''),\\ \Phi_1(k,k',k'') &= +\Phi(k',k'',k),\\ \Phi_2(k,k',k'') &= -\Phi(k'',k,k'),\\ \text{and}\qquad \Psi(k,k',k'') &= \Psi(k',k'',k) = -\Psi(k',k,k''),\end{aligned}\right\} \tag{51}$$

† In the present context, this is the analogue of the 'gauge-invariance' considered by Chandrasekhar (1950).

where the defining scalars in equation (50) are all regarded as being written with their independent variables in the order k, k', k''. Thus there are only *two* independent third-order velocity-product mean values in isotropic turbulence—a surprisingly small number.

Finally, it may be noted that the reality conditions (18) and (19) show that all the defining scalars in equations (48) and (50) are real functions.

Having thus obtained an explicit representation of the basic spectral tensors of isotropic turbulence in terms of defining scalars, we are now in a position to proceed with the derivation of the dynamical equations for these scalars.

4. Dynamical equations in isotropic turbulence

The well-known equation for the single defining scalar of $\Phi_{ij}(\mathbf{k})$ is most easily found by contracting the indices in equation (33). Thus, writing the resulting equation in the usual form

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k), \tag{52}$$

the expression for the function $T(k)$, which represents the rate of change of the energy spectrum due to the inertial transfer of energy between different wave-numbers, is

$$T(k) = 4\pi k^2 k_l \int \Phi_{iil}(\mathbf{k}, \mathbf{k}')\, \mathrm{d}\mathbf{k}'. \tag{53}$$

In terms of the defining scalars of $\Phi_{ijk}(\mathbf{k}, \mathbf{k}')$, this last result becomes

$$T(k) = \frac{\pi}{4}\int \frac{Q}{k'^2 k''^2}\left[2\{k^2(k''^2 - 3k'^2) - (k''^2 - k'^2)^2\}\,\Phi(k, k', k'') + 2k''^2(k''^2 - k^2)\,\Phi(k', k'', k) - k''^2 Q \Psi(k, k', k'')\right] \mathrm{d}\mathbf{k}', \tag{54}$$

where Q is the symmetric quartic

$$Q = k^4 + k'^4 + k''^4 - 2k^2k'^2 - 2k'^2k''^2 - 2k''^2k^2.$$

The expression (54) for the transfer function does not contain any approximation.

Consider next the equation (42) for $\Phi_{ijk}(\mathbf{k}, \mathbf{k}')$. In terms of the vector $\mathbf{k}''$, the operator Ω_3 is (cf. equation (39))

$$\Omega_3\{\Gamma_{ijk}(\mathbf{k}, \mathbf{k}')\} = \Gamma_{ijk}(\mathbf{k}, \mathbf{k}') + \Gamma_{jki}(\mathbf{k}', \mathbf{k}'') + \Gamma_{kij}(\mathbf{k}'', \mathbf{k}),$$

so that equation (42) may be written

$$\frac{\partial}{\partial t}\Phi_{ijk}(\mathbf{k}, \mathbf{k}') = \Omega_3\{k''_l \Delta_i^{\alpha}(\mathbf{k}'')\,[\Phi_{\alpha j}(\mathbf{k})\,\Phi_{lk}(\mathbf{k}') + \Phi_{\alpha k}(\mathbf{k}')\,\Phi_{lj}(\mathbf{k})]\} - \nu(k^2 + k'^2 + k''^2)\,\Phi_{ijk}(\mathbf{k}, \mathbf{k}'). \tag{55}$$

Now the transformation of this last equation into an equivalent set of scalar equations may be effected by writing every term in the orthogonal form (50) and equating the coefficients of like tensors. The results for the term on the left and the second term on the right of equation (55) are obvious, and it only remains to consider the first term on the right.

Since the tensor under the operator Ω_3 in equation (55) possesses the same orthogonality properties as $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$, namely, those set out in (24), it too may be represented in the manner (50). Thus, substituting the isotropic forms of $\Phi_{ij}(\mathbf{k})$ and $\Phi_{ij}(\mathbf{k}')$, one finds

$$k_l''\Delta_i^\alpha(\mathbf{k}'')\,[\Phi_{\alpha j}(\mathbf{k})\,\Phi_{lk}(\mathbf{k}')+\Phi_{\alpha k}(\mathbf{k}')\,\Phi_{lj}(\mathbf{k})]$$
$$=\Delta_i^\alpha(\mathbf{k}'')\,\Delta_j^\beta(\mathbf{k})\,\Delta_k^\gamma(\mathbf{k}')\frac{E(k)\,E(k')}{16\pi^2k^2k'^2}(k'_\beta\delta_{\gamma\alpha}+k_\gamma\delta_{\alpha\beta}). \tag{56}$$

The final step consists of performing the operation Ω_3 on the right-hand side of equation (56); this yields the result

$$\Delta_i^\alpha(\mathbf{k}'')\,\Delta_j^\beta(\mathbf{k})\,\Delta_k^\gamma(\mathbf{k}')\frac{1}{16\pi^2}\Big[\frac{E(k'')}{k''^2}\Big\{\frac{E(k')}{k'^2}-\frac{E(k)}{k^2}\Big\}k_\alpha\delta_{\beta\gamma}$$
$$+\frac{E(k)}{k^2}\Big\{\frac{E(k'')}{k''^2}-\frac{E(k')}{k'^2}\Big\}k'_\beta\delta_{\gamma\alpha}+\frac{E(k')}{k'^2}\Big\{\frac{E(k'')}{k''^2}-\frac{E(k)}{k^2}\Big\}k_\gamma\delta_{\alpha\beta}\Big]. \tag{57}$$

Every term in equation (55) is now in the required form, and the dynamical equations for the defining scalars of $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ may be written down immediately. The two independent equations are

$$\frac{\partial}{\partial t}\Phi(k,k',k'')=\frac{E(k'')}{16\pi^2k''^2}\Big[\frac{E(k')}{k'^2}-\frac{E(k)}{k^2}\Big]-\nu(k^2+k'^2+k''^2)\,\Phi(k,k',k'') \tag{58}$$

and

$$\frac{\partial}{\partial t}\Psi(k,k',k'')=-\nu(k^2+k'^2+k''^2)\,\Psi(k,k',k''). \tag{59}$$

These equations, together with (52) and (54), form the basis of all further results concerned specifically with isotropic turbulence.

PART II. ON THE DECAY OF ISOTROPIC TURBULENCE AT LARGE REYNOLDS NUMBERS

It will be clear from the complexity of the governing equations, even in the relatively simple case of isotropic turbulence, that any attempt to determine the detailed course of events following upon given initial conditions must necessarily involve a considerable amount of numerical integration. Moreover, no really adequate assessment of the value of the ideas developed in this paper can be made without recourse to such detailed numerical methods. Nevertheless, it is possible to obtain by analytical methods one or two results of a more general character that give some indication of the processes taking place. These results, which refer to a field of isotropic turbulence in which inertia forces are dominant, are presented in the following sections.

5. A dynamical property of $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$

A curious feature of the dynamical equations (58) and (59) is the existence of two obvious time integrals. The solutions in question are obtained from the fact that the quantities $\Psi(k,k',k'')$ and

$$\Sigma(k,k',k'')=\Phi(k,k',k'')+\Phi(k',k'',k)+\Phi(k'',k,k') \tag{60}$$

both decay from their initial values by viscous action alone, and consequently have the simple forms

$$\Psi(k,k',k'',t) = \Psi(k,k',k'',t_0)\,\mathrm{e}^{-\nu(k^2+k'^2+k''^2)(t-t_0)} \tag{61}$$

and

$$\Sigma(k,k',k'',t) = \Sigma(k,k',k'',t_0)\,\mathrm{e}^{-\nu(k^2+k'^2+k''^2)(t-t_0)}. \tag{62}$$

Apparently the interaction between $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ and $\Phi_{ij}(\mathbf{k})$ is such that only certain components of the former tensor are affected, the remainder dying away in the manner of viscous transients.

Unfortunately, in order to express this information in terms of the transfer function $T(k)$, it is necessary to perform the integration (54) over all $\mathbf{k}'$-space. The symmetry of the solution is then largely lost, and we have not been able to find any simple physical interpretation. If the initial Reynolds number is at all large, however, the time scale of decay of the transients (61) and (62) must also be large, and this raises interesting possibilities concerning the length of the period of decay of the turbulence during which inertia forces are appreciable.

6. The production of vorticity by the strain field

When the Reynolds number is large, the mechanism which dominates the decay process as a whole is the transfer of energy from large to small eddies. Now, unlike most contributions to the theory of decay, the assumptions underlying the present work do not make any explicit reference to energy transfer, and it is by no means obvious that these assumptions lead to a transfer mechanism which is physically acceptable even in its most elementary essentials. It is not clear, for instance, whether or not energy is transferred in the right direction, and it is important that such a general question as this should be settled.

Now, there exists a remarkably simple exact solution of the general inviscid equations which is relevant to this problem. The solution in question concerns the integral

$$\int_0^\infty k^2 E(k)\,\mathrm{d}k,$$

which is proportional to the mean-square vorticity of the turbulence. According to (52), the inviscid rate of change of this quantity is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty k^2 E(k)\,\mathrm{d}k = \int_0^\infty k^2 T(k)\,\mathrm{d}k, \tag{63}$$

and the physical mechanism which (63) describes is the stretching of vortex lines by the local strain field. For a transfer of energy from large to small eddies, the terms in (63) are positive, and the average effect of the strain is an extension rather than a compression of the vortex lines.

A second relation between the integrals occurring in (63) may be obtained by considering the inviscid rate of change of the transfer function $T(k)$. Thus, differentiating with respect to time the general expression (54) for this function, and substituting the rates of change

$$\frac{\partial}{\partial t}\Phi(k,k',k'') = \frac{E(k'')}{16\pi^2 k''^2}\left[\frac{E(k')}{k'^2} - \frac{E(k)}{k^2}\right]$$

and

$$\frac{\partial}{\partial t}\Psi(k,k',k'') = 0$$

in the integrand, one obtains after some reduction, the result

$$\frac{\partial T(k)}{\partial t} = -\frac{E(k)}{12k^2}\int_0^\infty (13k^4 - 8k^2k'^2 + 3k'^4)\, E(k')\,\mathrm{d}k' - \frac{E(k)}{8k^3}\int_0^\infty \frac{(k^2-k'^2)^3}{k'} \ln\left(\frac{k+k'}{|k-k'|}\right) E(k')\,\mathrm{d}k'$$
$$-\frac{1}{8k}\int_0^\infty \frac{E(k')}{k'^3}\int_{|k-k'|}^{k+k'} (k^2-k'^2+k''^2)\, Q\frac{E(k'')}{k''}\,\mathrm{d}k''\,\mathrm{d}k'. \quad (64)$$

If (64) is now multiplied by k^2 and integrated over all values of k, there results the very simple equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty k^2 T(k)\,\mathrm{d}k = \frac{2}{3}\left[\int_0^\infty k^2 E(k)\,\mathrm{d}k\right]^2. \quad (65)$$

Combining (63) and (65), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_0^\infty k^2 E(k)\,\mathrm{d}k = \frac{2}{3}\left[\int_0^\infty k^2 E(k)\,\mathrm{d}k\right]^2. \quad (66)$$

Alternatively, if $\overline{\omega^2}$ denotes the mean-square value of one component of vorticity, then

$$\overline{\omega^2} = \frac{2}{3}\int_0^\infty k^2 E(k)\,\mathrm{d}k,$$

so that (66) may be written

$$\frac{\mathrm{d}^2\overline{\omega^2}}{\mathrm{d}t^2} = (\overline{\omega^2})^2. \quad (67)$$

For the type of probability distribution under discussion, this last equation is the statistical analogue of Cauchy's equation in a frictionless fluid. One integration gives

$$\left(\frac{\mathrm{d}\overline{\omega^2}}{\mathrm{d}t}\right)^2 = \tfrac{2}{3}\,[(\overline{\omega^2})^3 - (\overline{\omega_0^2})^3], \quad (68)$$

where $\overline{\omega_0^2}$ is a constant, and the solution of this last equation is the Weierstrassian elliptic function

$$\overline{\omega^2} = 2^{\frac{2}{3}}\overline{\omega_0^2}\,\wp(x;\ 0, 1), \quad (69)$$

where, with a suitable choice of the origin of time,

$$x = 2^{\frac{1}{3}}(\tfrac{1}{6}\overline{\omega_0^2})^{\frac{1}{2}}\, t.$$

In the physical problem, only one real period of the doubly periodic elliptic function is relevant, i.e. $0 \leqslant x \leqslant 2x_0$, where $x_0 \doteqdot 1{\cdot}53$. The solution† then has the form shown in figure 1.

Provided that viscous effects are neglected, the slope of the curve in figure 1 is proportional to

$$\int_0^\infty k^2 T(k)\,\mathrm{d}k,$$

and the sign of this slope is an overall measure of the direction of energy transfer. The solution then shows that the only circumstances in which there is a general transfer of energy from small to large eddies are those in which this situation is prescribed as part of the initial conditions, i.e. when the initial conditions correspond to a point in the left-hand half of the figure. Moreover, even under these circumstances, the subsequent effect of inertia forces is to reduce the level of transfer and eventually change its direction. Irrespective of the initial conditions, therefore, after a finite time the direction of the transference of energy is always from large to small eddies.

† Tables of $\wp(x;\ 0, 1)$, the so-called equi-anharmonic case, are given in Jahnke & Emde's *Tables of functions*.

The asymptotic behaviour of the vorticity may easily be found from (69). Thus, if t_0 is the time corresponding to $x \doteqdot 2x_0$, then as $t \to t_0$ the vorticity becomes infinite according to the universal law

$$\overline{\omega^2} = \frac{6}{(t_0-t)^2} + O\{(t_0-t)^4\}, \tag{70}$$

which is completely independent of the initial conditions. As is to be expected, however, the time scale of the achievement of this asymptotic state does depend on the initial conditions, since a measure of this scale is

$$t_0 = \frac{2x_0}{2^{\frac{1}{3}}(\frac{1}{6}\overline{\omega_0^2})^{\frac{1}{2}}} \doteqdot \frac{5\cdot 9}{(\overline{\omega_0^2})^{\frac{1}{2}}}. \tag{71}$$

These results refer, of course, to a frictionless fluid. In a real fluid, the infinite vorticity is prevented by the diffusive properties of the viscosity.

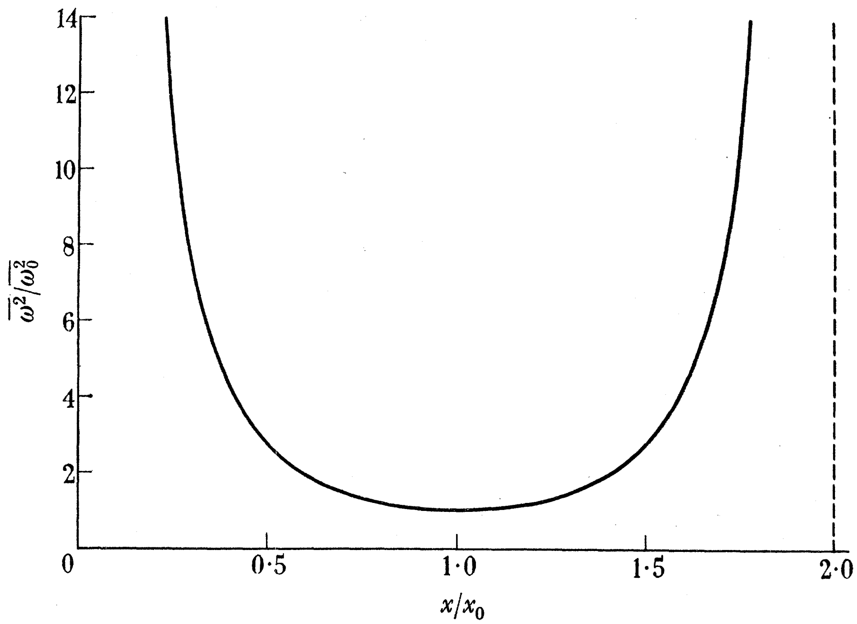

FIGURE 1. The production of vorticity

It will be noticed that the behaviour of the vorticity described above is in accordance with the predictions of Kolmogoroff's theory of local similarity. According to that theory, the skewness factor S of the probability distribution of $-\partial u_1/\partial x_1$ is a universal constant for fields of turbulence with sufficiently large Reynolds numbers. Hence, from the usual (exact) formula

$$\frac{\mathrm{d}\overline{\omega^2}}{\mathrm{d}t} = \frac{7S}{3\sqrt{5}}(\overline{\omega^2})^{\frac{3}{2}}, \tag{72}$$

for the production of vorticity by the stretching of vortex lines, it follows that

$$\left(\frac{\mathrm{d}\overline{\omega^2}}{\mathrm{d}t}\right)^2 \propto (\overline{\omega^2})^3,$$

which is in agreement with (68) when the time of decay is great enough for the asymptotic state (70) to have been established.

It is, of course, true that the quantities considered in this section are predominantly determined by eddy sizes much smaller than those in which the underlying approximation in the probability distribution receives experimental support. Indeed, the whole concept of a normal probability distribution is quite foreign to the statistical equilibrium of the smaller eddies. The former property of the motion, when it exists, must in large measure be due to the survival of initial circumstances which the mechanical processes are unable to destroy, whereas the latter property is entirely brought about by these same processes. But it is just for this reason that the results have seemed worth while presenting. The assumed relation between fourth- and second-order velocity-product mean values may be regarded as a constraint on the small eddies, and it is an indication of the remarkable universality of Kolmogoroff's ideas that, despite such an unsympathetic constraint, at least some aspects of the small eddy motion inevitably approach the predicted asymptotic form irrespective of the initial conditions.

At the same time, it is perhaps permissible to deduce that the properties of the present model of turbulence at large wave-numbers are sufficiently realistic to ensure that no great error will arise from this source in a discussion of the main energy-containing range of the spectrum. A concrete illustration of this is the estimate of S obtained from (68) and (72); for the theoretical value 0·78 compares quite favourably, under the circumstances,† with the observed value of about 0·35.

7. An example of the distribution of energy transfer

The results of the preceding section were concerned with the integral properties of energy transfer and yielded no information about the distribution of this quantity throughout the spectrum. However, it is possible to throw some light on the latter problem by calculating the inviscid rate of change of the function $T(k)$ according to equation (64), when some simply shaped energy spectrum is substituted in the right-hand side of that equation. For the sake of definiteness, this calculation may be considered as giving the initial rate of growth of the function $T(k)$ from a prescribed initial value of zero, when the given form of $E(k)$ represents the initial distribution of energy.

Now, a suitable choice of spectrum for this purpose is

$$E(k) = E_0 x^4 \,\mathrm{e}^{-x^2}, \quad x = k/k_0, \tag{73}$$

where E_0 and k_0 are parameters independent of k. The energy distribution (73) is typical of the purely viscous mechanism in the final period of decay of turbulence. If, therefore, the parameters E_0 and k_0 are so chosen that the Reynolds number is in fact large, the region of viscous dissipation of energy is removed to relatively large wave-numbers, and one would expect a transfer mechanism to develop of the kind that is observed in practice. More specifically, one would expect the general features of the initial distribution of $\partial T(k)/\partial t$ to be somewhat similar to those observed in the function $T(k)$ itself.

For the spectrum (73), the formula (64) gives

$$\left[\frac{\partial T(k)}{\partial t}\right]_{t=0} = k_0^3 E_0^2 \sqrt{\pi}\left[\tfrac{1}{4}x^2(-12+5x^2-2x^4)\,\mathrm{e}^{-x^2} + \tfrac{3}{4}x(4+x^2)\,\mathrm{e}^{-2x^2}\int_0^x \mathrm{e}^{\xi^2}\,\mathrm{d}\xi + \frac{1}{64\sqrt{2}}x^4(7+x^2)\,\mathrm{e}^{-\frac{1}{2}x^2}\right], \tag{74}$$

† We should also point out that the theory contains no adjustable parameters.

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which is shown graphically in figure 2. Also included in the figure are the energy and energy-dissipation curves, $E(k)$ and $k^2E(k)$, for the purpose of providing dynamical wave-number standards. Relative to the energy spectrum, the distribution of the loss of energy by the energy-containing eddies of this particular field of turbulence appears to possess many of the properties of typical distributions determined experimentally. Relative to the dissipation spectrum, however, the distribution of the transfer function is far from being in accordance with observations at large Reynolds numbers. But such is to be expected; experimental measurements are always made under conditions in which the smaller eddies have reached some type of approximate equilibrium, and the example (74) merely represents the initial movement towards such a state. Only the first-mentioned range of wave-numbers is suitable† for comparison with experiment, and inasmuch as a single example may be taken as evidence the qualitative features of the distribution are encouranging.

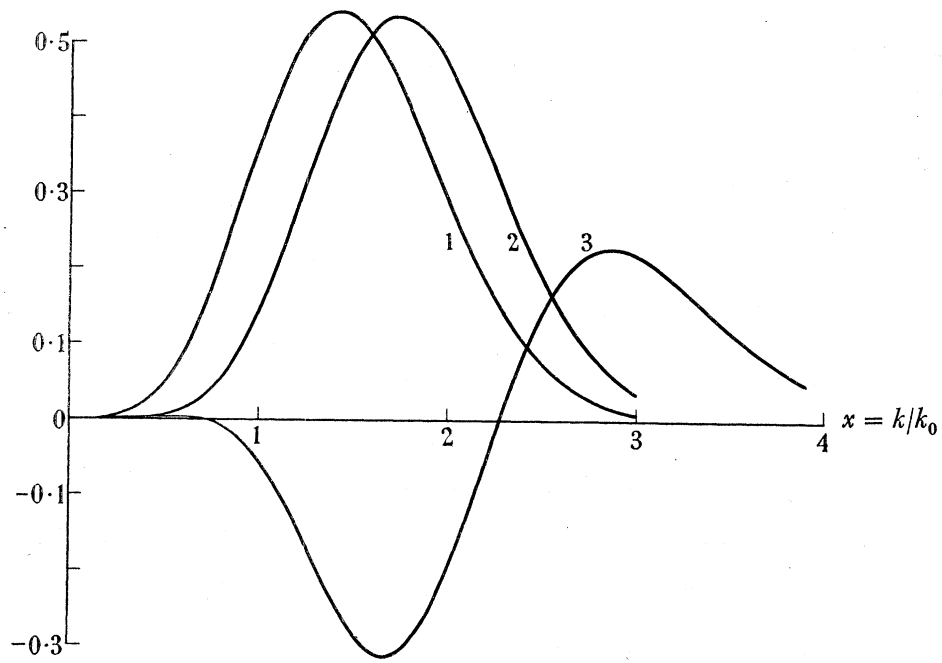

FIGURE 2. An example of the rate of growth of energy transfer. Curve 1, the energy spectrum, $E(k)/E_0$; curve 2, the energy-dissipation spectrum, $0{\cdot}4k^2E(k)/k_0^2E_0$; curve 3, $[\partial T(k)/\partial t]_{t=0}/k_0^3E_0^2$.

The magnitude of the rate of change (74) is also of interest. In the early stages of the decay of this field of turbulence, the transfer function is given by the expansion

$$T(k) = t\left[\frac{\partial T(k)}{\partial t}\right]_{t=0} + O(t^3), \tag{75}$$

and the time required for $T(k)$ to grow, according to (75), to a value comparable with those observed in practice is clearly an important index of the role likely to be played by inertia forces throughout the subsequent course of the decay. The comparison with experiment is best considered in terms of the triple-velocity correlation coefficient $k(r)$ introduced by

† Quite apart, that is, from the further reason that this is the only range in which the underlying approximation receives experimental support.

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von Kármán & Howarth (1938), since direct measurements of this function have recently been made by Stewart (1951). In the notation of § 1, $k(r)$ is defined by the equations

$$(\overline{u_1^2})^{\frac{3}{2}}\, k(r) = R_{111}(0, \mathbf{r}), \quad \mathbf{r} = (r, 0, 0), \tag{76}$$

and it may be shown from the definitions of the Fourier transform (15) and the transfer function (53) that

$$\frac{1}{2}\left(r\frac{\partial}{\partial r}+3\right)\left(\frac{\partial}{\partial r}+\frac{4}{r}\right)(\overline{u_1^2})^{\frac{3}{2}}\, k(r) = \int_0^\infty T(k)\,\frac{\sin kr}{kr}\,\mathrm{d}k \tag{77}$$

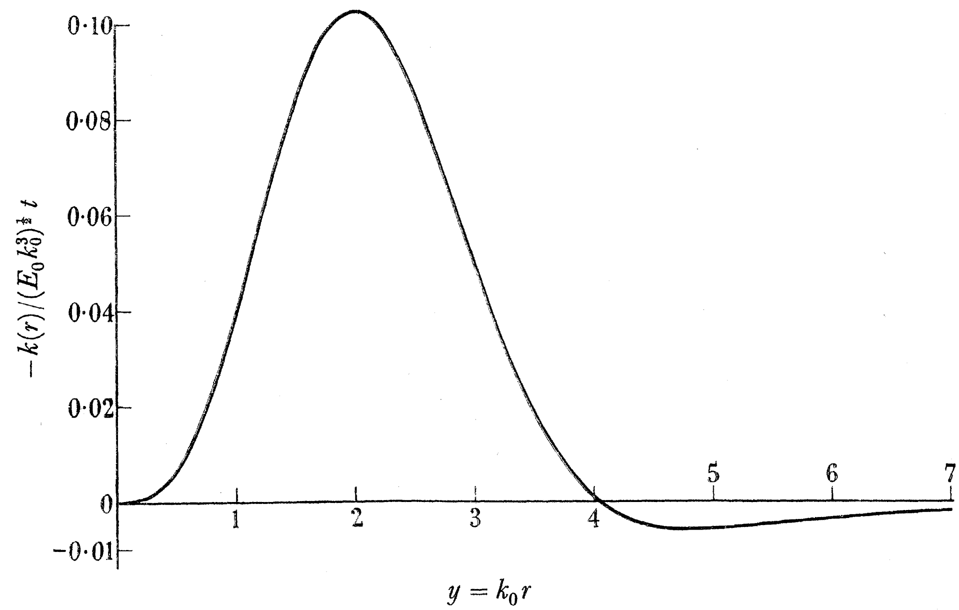

FIGURE 3. The triple correlation function corresponding to figure 3.

(see, for example, Batchelor 1953, p. 101). Hence, substituting the expression (74) in the right-hand side of (77) one obtains after some algebra the result†

$$k(r) = (E_0 k_0^3)^{\frac{1}{2}}\, t\,\frac{\pi^{\frac{1}{4}}}{8y^4}\Big[21\sqrt{(\tfrac{1}{2}\pi)}\,\operatorname{erf}\frac{y}{\sqrt{2}} + 12\sqrt{(2\pi)}\,(52-y^2)\,\mathrm{e}^{-\frac{1}{8}y^2}\operatorname{erf}\frac{y}{2\sqrt{2}}$$
$$-4y(156+10y^2+y^4)\,\mathrm{e}^{-\frac{1}{4}y^2} - y(21+7y^2-y^4)\,\mathrm{e}^{-\frac{1}{2}y^2}\Big] + O(t^3), \tag{78}$$

where $y = k_0 r$. This function is shown graphically in figure 3.

The figure shows the theoretical maximum value of $-k(r)/(E_0 k_0^3)^{\frac{1}{2}}\, t$ to be about 0·10, whereas the maximum value of $-k(r)$ obtained from Stewart's measurements in the initial period of decay is about 0·05. Hence, the requisite time for growth is

$$t \doteqdot \tfrac{1}{2}(E_0 k_0^3)^{-\frac{1}{2}}. \tag{79}$$

This estimate is of the order of the characteristic period of the energy-containing eddies of the spectrum (73), as is obviously demanded by the nature of the problem. What is relevant to the present calculation, however, is the value of the numerical coefficient in (79), for it appears that the experimental evidence sets fairly narrow limits to this number. Thus, it is

† A similar calculation was performed by Ellison (1952).

easily found that the proportion of the total energy lost by viscous dissipation (on the basis of the initial rate of change) during the interval of time (79) is

$$\frac{\overline{u_1^2}(0)-\overline{u_1^2}(t)}{\overline{u_1^2}(0)} \doteq \frac{2\cdot 4}{R_\lambda}, \tag{80}$$

where R_λ is the initial value of the Reynolds number of turbulence usually denoted by this symbol. For values of R_λ greater than about 20, therefore, the turbulence may be expected to become fully developed in the sense characterized in practice by the initial period of decay. For values of R_λ as small as 5, on the other hand, a substantial proportion of the turbulent energy would be lost before the transfer process could attain its fully developed level, and the decay would be more of the purely viscous type. This estimate of the 'critical' range of Reynolds numbers is very approximate, partly because only the initial rates of change are employed and partly because the result will depend to some extent on the initial choice of spectrum. Nevertheless, it is interesting to note that much the same estimate was deduced by Batchelor & Townsend (1948) from experimental results, and that a change in the coefficient in (79) by a factor of 5 or so would destroy the agreement to a noticeable extent.

A further feature of the example (74) that is worth noting is the distribution of energy transfer at small wave-numbers. Thus if this expression is expanded in a power series at the origin, one finds that

$$\left[\frac{\partial T(k)}{\partial t}\right]_{t=0} = \frac{7\pi}{128} E_0^2 k_0^3 x^4 + O(x^6). \tag{81}$$

Hence, the leading terms in the expansions of the energy spectrum and transfer function in this example are of the same order in k, and this is contrary to existing notions about the permanence of the large-scale components of a homogeneous turbulent motion. It is true, of course, that there is very little energy involved in this region of the spectrum, and that the effect on the decay process as a whole is negligible when the Reynolds number is large. But the phenomenon, which turns out to be of considerable generality, has obvious interest and is discussed in part III.

PART III. ENERGY TRANSFER AT SMALL WAVE-NUMBERS

The behaviour of the spectrum tensor $\Phi_{ij}(\mathbf{k})$ and the energy-spectrum function $E(k)$ at small wave-numbers, with which the large-scale components of the turbulence are associated, was considered first by Lin (1947) for isotropic turbulence and later by Batchelor (1949) for homogeneous turbulence. The description of the motion of these large-scale components given by these writers is that they are invariant throughout the entire decay process. As the example of the previous section indicates, however, the predictions of the present theory differ substantially from those of Lin and Batchelor. In the following sections, therefore, we present a further study of the whole problem of energy transfer at small wave-numbers, together with an examination of some of the relevant experimental data.

8. Isotropic turbulence

For simplicity, we consider first the behaviour of $\partial T(k)/\partial t$ for small values of k in isotropic turbulence. Since viscous effects are negligible at small wave-numbers unless the Reynolds number is very small, it is permissible to begin with the inviscid rate of change (64). Thus, the assumption that that equation may be expanded in powers of k leads to the result

$$\frac{\partial T(k)}{\partial t} = \tfrac{14}{15} k^4 \int_0^\infty \frac{E^2(k')}{k'^2}\,\mathrm{d}k' + O(k^6). \tag{82}$$

Now the integral occurring in this equation is quadratic in the energy spectrum and is therefore non-negative whatever the form of $E(k)$ and vanishes if, and only if, $E(k)$ is itself identically zero. This shows that the rate of change of the transfer function at small values of k is of order k^4 for an arbitrary distribution of energy. It then follows that $T(k)$ must itself be of order k^4, since, by the incompressibility condition, it cannot be of lower order. A further consequence of the continuity condition at small wave-numbers is that the energy spectrum $E(k)$ is also of order k^4; the behaviour of the spectrum in this region is described, therefore, by the equations

$$E(k) = C(t)\,k^4 + O(k^6) \tag{83}$$

and

$$\frac{\mathrm{d}^2 C(t)}{\mathrm{d}t^2} = \frac{14}{15} \int_0^\infty \frac{E^2(k)}{k^2}\,\mathrm{d}k. \tag{84}$$

Hence, for the type of probability distribution under discussion, the coefficient of the leading term in the expansion of $E(k)$ cannot be an invariant of the motion.

Before examining the discrepancy between these results and those of Lin (1947) concerning the permanence of the large-scale components of the motion, it is useful to consider their significance in physical space. This may be done by employing the function

$$K(r) = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) (\overline{u_1^2})^{\frac{3}{2}}\, k(r), \tag{85}$$

where $k(r)$ is the usual triple correlation defined by equation (76). Thus, from equation (77), the relation between this function and $T(k)$ is

$$T(k) = \frac{1}{\pi} \int_0^\infty K(r)\,(kr \sin kr - k^2 r^2 \cos kr)\,\mathrm{d}r. \tag{86}$$

Now, if $T(k)$ is expanded in terms of the moments of $K(r)$, then

$$T(k) = \frac{1}{3\pi} k^4 \int_0^\infty r^4 K(r)\,\mathrm{d}r + O(k^6), \tag{87}$$

and this expansion is consistent with equation (82) if, and only if,

$$\int_0^\infty r^4 K(r)\,\mathrm{d}r \neq 0. \tag{88}$$

The meaning of this integral in terms of the triple correlation $k(r)$ is easily seen by substituting for $K(r)$ from equation (85); a partial integration of (88) then gives

$$\int_0^\infty r^4 K(r)\,\mathrm{d}r = (\overline{u_1^2})^{\frac{3}{2}} \,[r^4 k(r)]_{r=\infty}, \tag{89}$$

so that, provided the condition (88) is satisfied,

$$k(r) = O(r^{-4}) \tag{90}$$

for large values of r.

Now this behaviour of $k(r)$ at large values of r is closely related to the dynamical behaviour of Loitsiansky's integral

$$\Lambda(t) = \int_0^\infty \overline{u_1^2} f(r)\, r^4 \,\mathrm{d}r, \tag{91}$$

where $f(r)$ is the usual longitudinal velocity correlation coefficient defined by the equations

$$\overline{u_1^2} f(r) = R_{11}(\mathbf{r}), \quad \mathbf{r} = (r, 0, 0); \tag{92}$$

for, if the dynamical equation for $f(r)$ (von Kármán & Howarth 1938),

$$\frac{\partial}{\partial t}\overline{u_1^2} f(r) = 2\nu\left(\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}\right)\overline{u_1^2} f(r) + K(r), \tag{93}$$

is multiplied by r^4 and integrated over all values of r, then one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \overline{u_1^2} f(r)\, r^4 \,\mathrm{d}r = \int_0^\infty r^4 K(r)\,\mathrm{d}r, \tag{94}$$

where, as usual, it has been assumed that $[r^4 \partial f(r)/\partial r]_{r=\infty} = 0$. This last equation may also be written in the form

$$\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t} = (\overline{u_1^2})^{\frac{3}{2}}\, [r^4 k(r)]_{r=\infty}, \tag{95}$$

whence, by equation (84),

$$\frac{\mathrm{d}^2\Lambda(t)}{\mathrm{d}t^2} = \tfrac{14}{5}\pi \int_0^\infty \frac{E^2(k)}{k^2}\,\mathrm{d}k, \tag{96}$$

which shows that, for the probability distribution under discussion, Loitsiansky's integral is not an invariant of the motion.

The relationship between the present work and that of Lin (1947), who found that $T(k)$ was of order k^6 for small values of k, is now clear. By assuming that $T(k)$ could be expanded in terms of the moments of $k(r)$, Lin obtained the result

$$T(k) = \frac{1}{15\pi} k^6 \int_0^\infty (\overline{u_1^2})^{\frac{3}{2}}\, k(r)\, r^5 \,\mathrm{d}r + O(k^8). \tag{97}$$

Now the essential assumption underlying this result is that the function $k(r)$ decreases sufficiently rapidly at large values of r for the integral in the leading term to exist. In the case of a normal probability distribution, however, the result (90) shows that this integral is infinite. By employing the function $K(r)$ in the expansion of $T(k)$, this particular difficulty is avoided, since the behaviour $k(r) \propto r^{-4}$ in no way influences the behaviour of $K(r)$ at large values of r.

Thus, there is a clear-cut discrepancy between equations (87) and (97). It is necessary therefore to examine the relative merits of the intuitive assumption that the triple correlation decreases faster than r^{-4} and the dynamical evidence, obtained by assuming a normal probability distribution, that it decreases like r^{-4}. Accordingly, we attempt, in the remaining paragraphs of this section, to assess so far as possible the extent to which the result (90) depends upon a normal probability distribution, and to examine the relevant experimental data.

For this purpose, it is convenient to consider first the general expansion of $T(k)$ at small values of k, without making any assumptions regarding the probability distribution of the velocity field. From equation (54) this expansion is found to be

$$T(k) = \tfrac{8}{15}\pi^2 k^4 \int_0^\infty \left\{ 28k'^2 \Phi(0,k',k') - 8k'^3 \left[\frac{\partial \Phi(k',k'',0)}{\partial k''}\right]_{k''=k'} - k'^4 \left[\frac{\partial^2 \Phi(k',k'',0)}{\partial k''^2}\right]_{k''=k'} \right\} \mathrm{d}k' + O(k^6). \quad (98)$$

All eddy sizes make a contribution to the coefficient of k^4 in (98), and so it is virtually certain that if this coefficient is to vanish at all times during decay it must do so for purely kinematical reasons. But the integral in (98) clearly does not vanish for such reasons, since in the special case of a normal probability distribution, which is certainly kinematically possible, its inviscid rate of change reduces to the non-zero form given by (82). We are led to conclude, therefore, that a transfer function of smaller order than k^4 for small values of k is extremely unlikely, and that the triple correlation $k(r)$ will usually be of order r^{-4} for large values of r. Apparently, intuitive reasoning about the rapidity of the tendency to statistical independence of conditions at widely separated points in the turbulence leads one astray for reasons that are intimately connected with the dynamical role of inertia forces.

Further support for the foregoing conclusion is provided by the viscous contribution to the rate of change of the transfer function at small wave-numbers. Thus, differentiating (98) with respect to time, and substituting the rate of change

$$\frac{\partial}{\partial t}\Phi(k,k',k'') = -\nu(k^2+k'^2+k''^2)\,\Phi(k,k',k''),$$

one finds

$$\frac{\partial T(k)}{\partial t} = -\tfrac{16}{15}\pi^2 \nu k^4 \int_0^\infty \left\{ 28k'^4 \Phi(0,k',k') - 10k'^5 \left[\frac{\partial \Phi(k',k'',0)}{\partial k''}\right]_{k''=k'} - k'^6 \left[\frac{\partial^2 \Phi(k',k'',0)}{\partial k''^2}\right]_{k''=k'} \right\} \mathrm{d}k' + O(k^6), \quad (99)$$

which is again of order k^4. Hence, even if the initial conditions are such that $T(k)$ is of order k^6 for small values of k, the action of viscosity alone is sufficient to ensure that this function will be of order k^4 at all subsequent times, unless further initial conditions are imposed on the turbulence. These further conditions are that all the integral moments of $\Phi(k,k',k'')$, like those occurring in equations (98) and (99) obtained by successively differentiating equation (98) with respect to time, must vanish initially. Such initial conditions would appear to be very special indeed.

There remains the question of whether the coefficient of k^4 in equation (82) provides a reasonable estimate of actual conditions when the Reynolds number is sufficiently large for the viscous contribution (99) to be negligible. In this connexion, it may be noted that the integral occurring in equation (82) is weighted fairly sharply in favour of a group of eddies somewhat larger than the energy-containing eddies of the turbulence; somewhat larger, in fact, than the eddies to which the experimental evidence in favour of a normal probability distribution directly relates. Thus, it is not possible to assess the validity of equation (82) with any certainty. If, however, the assumed form of the probability distribution is valid for these larger eddies, though not necessarily for the very large components in which $E(k)$ is proportional to k^4, then the equation should be substantially correct; otherwise, the coefficient of k^4 may possibly be in error by as much as an order of magnitude.

In view of the foregoing theoretical suggestions regarding the behaviour of the large-scale components of the motion, it is of interest to examine the relevant experimental data. Unfortunately, the experimental results (Stewart & Townsend 1951) relevant to the prediction that $k(r) = O(r^{-4})$ for large values of r are inconclusive; for, if a law of the form r^{-n} is fitted to the available data, it is found that n only reaches a value of about 1·8 at the largest

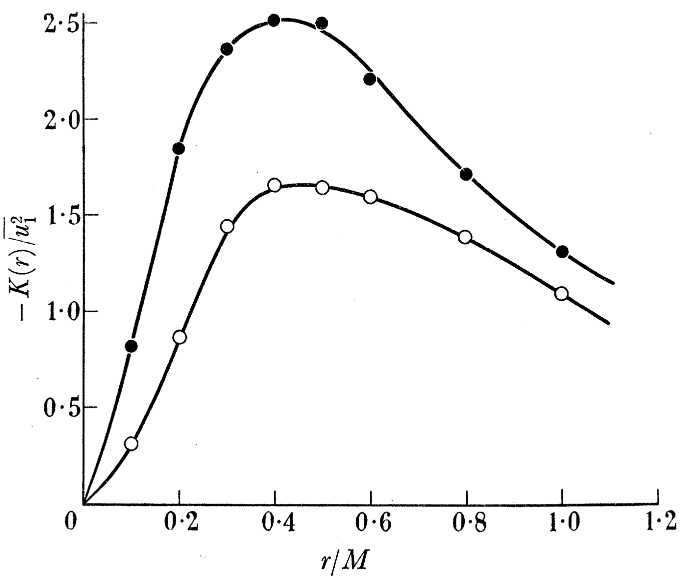

FIGURE 4. The triple correlation function $K(r)$ during decay of turbulence at $R_M = 5300$. x/M: ●, 60; ○, 90.

values of r for which measurements have been made. An alternative presentation of these results may be made by computing the function $K(r)$. For a different purpose, this calculation was carried out by Stewart (1951), and the curve in figure 4 is based upon his values. In this and subsequent figures referring to experimental data, x is the distance downstream from the grid producing the turbulence, and $R_M = UM/\nu$, where M is the grid spacing and U is the velocity of the mainstream. It may be noted that $K(r)$ does not change sign, so that the condition

$$\int_0^\infty r^4 K(r)\,\mathrm{d}r = 0,$$

which is necessary if $k(r)$ decreases more rapidly than r^{-4}, though not precluded, is hardly suggested by the available experimental evidence.

The question now arises as to whether the first few integral moments of $K(r)$ itself are finite, for this has been assumed in expanding $T(k)$ in the form (87). Here again, the direct experimental evidence is inconclusive. However, if the behaviour of the correlation function $f(r)$ is such that its integral moments of order up to and including Loitsiansky's integral are finite, then the dynamical equation (93) ensures that the corresponding moments of $K(r)$

exist. Thus, a sufficient condition for the validity of the leading term of the expansion (87) is simply the existence of Loitsiansky's integral. There is, in fact, a striking difference between the measurements of $f(r)$ and $K(r)$ on the one hand, and $k(r)$ on the other. This is displayed in figures 5 and 6, which are again based on the measurements of Stewart (1951). In view of the preceding analysis, the slow decrease of $k(r)$ compared with that of $K(r)$ and $f(r)$ is very suggestive.

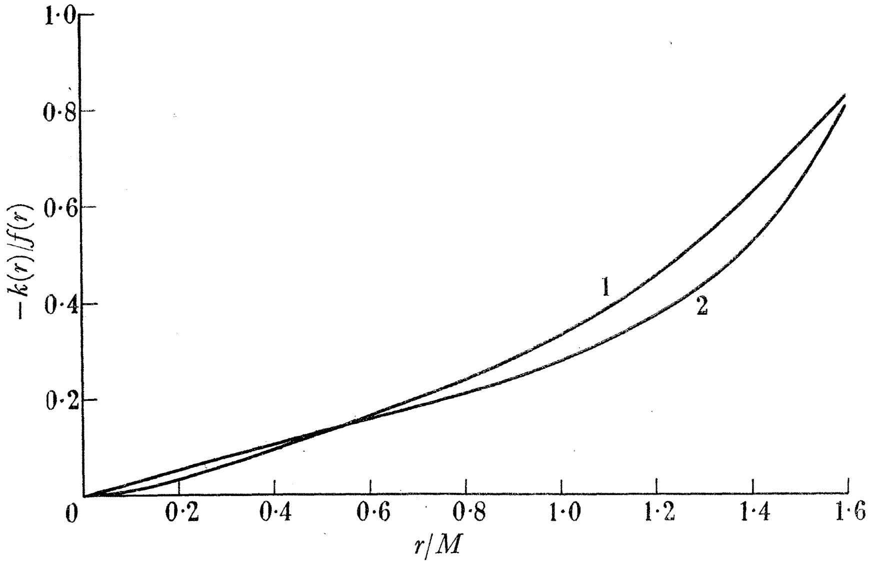

FIGURE 5. $x/M = 30$. Curve 1, $R_M = 5300$; curve 2, $R_M = 42000$.

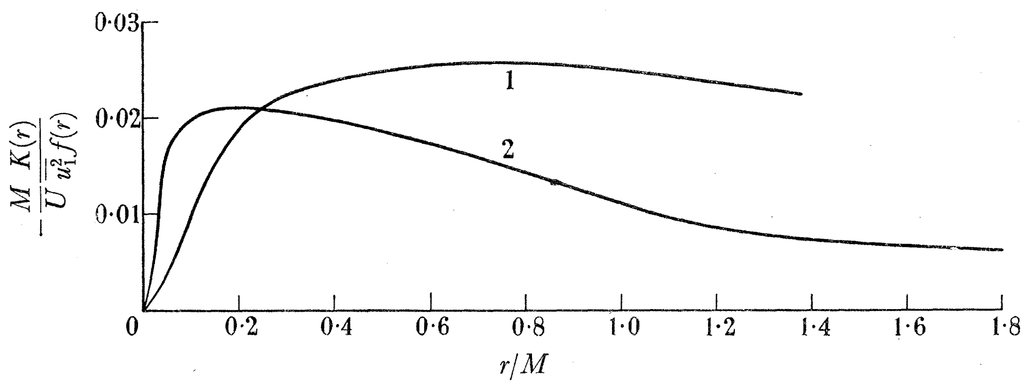

FIGURE 6. $x/M = 30$. Curve 1, $R_M = 5300$; curve 2, $R_M = 42000$.

9. HOMOGENEOUS TURBULENCE

While the study of energy transfer at small wave-numbers in isotropic turbulence introduces the problem in its simplest context, nevertheless, it is of considerable interest to extend these results as far as possible to the more general case of homogeneous turbulence. This more general aspect of the problem has already been considered in some detail by Batchelor (1949). But, since Batchelor's results reduce to Lin's in the special case of isotropic turbulence, it follows that there must be a fundamental discrepancy between the basic assumptions employed by Batchelor and those of the present analysis. The primary purpose of this section is to examine the nature of this discrepancy.

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In order to exhibit as clearly as possible both the similarities and the differences between the present theory and Batchelor's results, consider first the dynamical equation for $\Phi_{ij}(\mathbf{k})$ in homogeneous turbulence, which may be written in the form (cf. equation (33))

$$\frac{\partial}{\partial t}\Phi_{ij}(\mathbf{k}) = \Sigma_{ij}(\mathbf{k}) - 2\nu k^2 \Phi_{ij}(\mathbf{k}), \tag{100}$$

where
$$\Sigma_{ij}(\mathbf{k}) = \mathrm{O}_2\left\{k_k\,\Delta_i^{\alpha}(\mathbf{k})\int \Phi_{\alpha jk}(\mathbf{k},\mathbf{k}')\,\mathrm{d}\mathbf{k}'\right\}. \tag{101}$$

The tensor $\Sigma_{ij}(\mathbf{k})$ plays essentially the same role in homogeneous turbulence that the transfer function $T(k)$ does in isotropic turbulence. Indeed, when the turbulence is isotropic $T(k)$ is the defining scalar of $\Sigma_{ij}(\mathbf{k})$ in exactly the same sense as $E(k)$ is the defining scalar of $\Phi_{ij}(\mathbf{k})$ (cf. equation (48)). Accordingly, in a discussion of energy transfer at small wave-numbers we are primarily interested in the behaviour of $\Sigma_{ij}(\mathbf{k})$ for small values of k.

In order to keep the analysis as simple as possible, it is assumed throughout this section that the probability distribution of the velocity field is normal; for, if the conclusions reached in the case of isotropic turbulence are valid, then it is clear that this assumption retains all the essential features of the problem. Moreover, in calculating the rate of change of $\Sigma_{ij}(\mathbf{k})$, we again assume the Reynolds number to be sufficiently large for viscous contributions of the type (99) to be negligible. Thus, if the inviscid rate of change of $\Phi_{ijk}(\mathbf{k},\mathbf{k}')$ (equation (42)) is substituted into the time derivative of equation (101), and $\partial\Sigma_{ij}(\mathbf{k})/\partial t$ is expanded in powers of the components of $\mathbf{k}$, one finds after some reduction that

$$\frac{\partial}{\partial t}\Sigma_{ij}(\mathbf{k}) = 2k_l k_m\,\Delta_i^{\alpha}(\mathbf{k})\,\Delta_j^{\beta}(\mathbf{k})\int[\Phi_{\alpha l}(\mathbf{k}')\,\Phi_{m\beta}(\mathbf{k}') + \Phi_{\alpha\beta}(\mathbf{k}')\,\Phi_{lm}(\mathbf{k}')]\,\mathrm{d}\mathbf{k}' + O(k^3), \tag{102}$$

and this result shows that $\partial\Sigma_{ij}(\mathbf{k})/\partial t$ is of order k^2 for small values of k.

On the other hand, Batchelor (1949) found that $\Sigma_{ij}(\mathbf{k}) = O(k^3)$, and hence that, in his expansion for the spectrum tensor

$$\Phi_{ij}(\mathbf{k}) = C_{ijlm}k_l k_m + O(k^3), \tag{103}$$

the coefficient C_{ijlm} is invariant throughout the entire decay process. The result (102) clearly does not permit such invariance, and it is necessary to compare the two analyses in greater detail. Batchelor works entirely in terms of the single vector $\mathbf{k}$ and the tensors†

$$\Upsilon_{ijk}(\mathbf{k}) = \int \Phi_{ijk}(\mathbf{k},\mathbf{k}')\,\mathrm{d}\mathbf{k}', \tag{104}$$

which is essentially the Fourier transform of $R_{ijk}(\mathbf{r},0) = \overline{u_i(\mathbf{x})\,u_j(\mathbf{x}+\mathbf{r})\,u_k(\mathbf{x})}$, and $\Pi_j(\mathbf{k})$, which, by the continuity condition, is related to $\Upsilon_{ijk}(\mathbf{k})$ in the manner (cf. equation (32))

$$k^2\Pi_j(\mathbf{k}) = k_i k_k\,\Upsilon_{ijk}(\mathbf{k}). \tag{105}$$

By assuming that both of these tensors could be expanded as power series in $\mathbf{k}$, Batchelor found, as a consequence of the incompressibility condition at small wave-numbers, that

$$\Pi_j(\mathbf{k}) = k_k\,\Pi_{jk} + \dots \quad \text{and} \quad \Upsilon_{ijk}(\mathbf{k}) = k_l\,\Upsilon_{ijkl} + \dots, \tag{106}$$

† The tensors $\Upsilon_{ijk}(\mathbf{k})$ and $\Pi_j(\mathbf{k})$ used here are denoted by $-\mathrm{i}\Upsilon_{ikj}(\mathbf{k})$ and $-\mathrm{i}\Theta_j(\mathbf{k})$ respectively by Batchelor (1953).

where the coefficients Π_{jk} and Υ_{ijkl} are completely independent of $\mathbf{k}$ and satisfy certain permutation relations. Then, by substituting these expansions into equation (105), he derived a further relationship between Π_{jk} and Υ_{ijkl} which is completely equivalent to the statement that Υ_{ijkl} must be expressible in the form

$$\Upsilon_{ijkl} = \epsilon_{jlm}[A_m \delta_{ik} + B_k \delta_{im} + B_i \delta_{km}], \tag{107}$$

where the vectors $\mathbf{A}$ and $\mathbf{B}$ are independent of $\mathbf{k}$ and ϵ_{ijk} is the alternating tensor. Now $\Sigma_{ij}(\mathbf{k})$ may be written in the form

$$\Sigma_{ij}(\mathbf{k}) = \mathrm{O}_2\{k_k \Delta_i^\alpha(\mathbf{k}) \Upsilon_{\alpha jk}(\mathbf{k})\}, \tag{108}$$

and when the explicit expansion of $\Upsilon_{ijk}(\mathbf{k})$ in terms of the vectors $\mathbf{A}$ and $\mathbf{B}$ is substituted into this equation one obtains the result found by Batchelor (1949), namely, that $\Sigma_{ij}(\mathbf{k})$ is of order k^3 for small values of k.

The discrepancy between this result and equation (102) may be traced to assumptions of the type (106), and is best illustrated by considering the special case of isotropic turbulence while still retaining the tensor form in the analysis. Thus, the tensor $\Upsilon_{ijk}(\mathbf{k})$ is orthogonal with respect to $\mathbf{k}$ in the index j and symmetric in the indices i and k; it may therefore be written in the form

$$\Upsilon_{ijk}(\mathbf{k}) = \Delta_j^\beta(\mathbf{k})\,(k_i \delta_{\beta k} + k_k \delta_{\beta i})\,\Upsilon(k), \tag{109}$$

where $\Upsilon(k)$ is an even scalar function of k which we suppose to possess an expansion of the form

$$\Upsilon(k) = \Upsilon(0) + \Upsilon''(0)\frac{k^2}{2} + O(k^4). \tag{110}$$

Now, by virtue of the non-analytic nature of the factor

$$\Delta_j^\beta(\mathbf{k}) = \delta_{j\beta} - k_j k_\beta / k^2$$

near $\mathbf{k} = 0$, Batchelor's assumption (106) that $\Upsilon_{ijk}(\mathbf{k})$ must be analytic in the individual components of $\mathbf{k}$ leads to the condition

$$\Upsilon(0) = 0, \tag{111}$$

and hence that

$$\Sigma_{ij}(\mathbf{k}) = (k^2\delta_{ij} - k_i k_j)\,\Upsilon(k) \tag{112}$$

is of order k^4. If, however, $\Upsilon(0)$ does not vanish, then $\Upsilon_{ijk}(\mathbf{k})$ is non-analytic and $\Sigma_{ij}(\mathbf{k})$ is clearly of order k^2.

Now, if the arguments presented in the preceding section are valid, a normal probability distribution of the velocity field may be regarded as 'typical' for the purpose in hand, and equation (102) then shows that $\Sigma_{ij}(\mathbf{k})$ is of order k^2 for small values of k. A comparison with Batchelor's results then indicates that $\Upsilon_{ijk}(\mathbf{k})$ cannot be an analytic function of $\mathbf{k}$. In this connexion, we should remark that a non-analytic behaviour of $\Upsilon_{ijk}(\mathbf{k})$ implies the non-existence of certain integral moments of $R_{ijk}(\mathbf{r}, \mathbf{r}')$, which agrees with the specific form of the difference between the present results and those of Lin in the case of isotropic turbulence.

There is, however, a further difficulty in homogeneous turbulence which does not arise in the isotropic case. Equation (112) shows that, in isotropic turbulence, $\Sigma_{ij}(\mathbf{k})$ is an analytic function even when $\Upsilon(0) \neq 0$. Thus, a non-analytic form of $\Upsilon_{ijk}(\mathbf{k})$ is dynamically consistent with an analytic spectrum tensor $\Phi_{ij}(\mathbf{k})$. This appears not to be the case in homogeneous turbulence; for, due to the presence of the factor $\Delta_i^\alpha(\mathbf{k})\,\Delta_j^\beta(\mathbf{k})$, the coefficient of $k_l k_m$ in equation (102) depends in general on the direction of $\mathbf{k}$, and this is inconsistent with a simple expansion of the type (103) assumed by Batchelor. Clearly, there is here a new type of difficulty which is necessarily encountered in any analysis that takes into account the dynamical equation for $\Phi_{ijk}(\mathbf{k}, \mathbf{k}')$. For, unless it can be demonstrated that the conditions under which $\Sigma_{ij}(\mathbf{k})$ is an analytic function are relatively mild, the conclusion seems inescapable that an analytic spectrum tensor $\Phi_{ij}(\mathbf{k})$ is dynamically impossible. In the case of a normal probability distribution, we have not been able to find such a demonstration.

In conclusion, we wish to thank Dr G. K. Batchelor, Dr T. H. Ellison and Dr A. A. Townsend for the many hours they have spent discussing with us the various problems examined in this paper.

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